Approximation Solution of Fredholm Integral Equation Using Adomian Decomposition Method

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Abstract

In this paper, Adomian Decomposition method has been used to find the approximation solution for the linear Fredholm integral equation of the second kind. In this method the solution of a functional equation is considered as the sum of an infinite series usually converging to the solution, and the Adomian decomposition method is also used to solve linear integral equation. Finally, numerical example are prepared to illustrate these considerations.

حل تقريبي لمعادلة فريدهوم التكاملية باستخدام طريقة ادموند الانحلالية

أمنة قاسم حسين
ماجستير في الرياضيات التطبيقية
كلية الهندسة - الجامعة المستنصرية

الخلاصة

في هذا البحث استخدمت طريقة الانحلال لإيجاد حل تقريبي لمعادلة فريدهوم التكاملية من النوع الثاني. والحل في هذه الطريقة يكون دالة تمثل مجموع متسلسلة غير محددة تتقارب إلى الحل. كما أن طريقة أدموند الانحلالية هي طريقة لحل المعادلات التكاملية الخطية وغير الخطية. والمثال المطبق يوضح هذه الطريقة.

1. Introduction

The integral equation is generally defined as an equation which involves the integral of an unknown function u(x) appears under the integral sign.

Therefore the general form of integral equation is given by the following form, [1]:

$$a(x)u(x) - \int_{\Omega} k(x,t)u(t)dt = f(x) \qquad \ldots(1)$$

where a, f and k are known functions; $k(x,t)$ is called the kernel of the integral equation, u is the unknown function to be determined and Ω be a measurable set in a measurable space E.

We can distinguish between two types of integral equations which are,[2]:

1. Integral equation of the first kind when $a(x) = 0$ in equation (1):

$$f(x) = \int_{\Omega} k(x,t)u(t)dt \qquad \ldots(2)$$

2. Integral equation of the second kind when $a(x) \neq 0$, then equation (1) can be written as:

$$u(x) = f(x) + \int_{\Omega} k(x,t)u(t)dt \qquad \ldots(3)$$

Now integral equations can be classified into different kinds according to the limits of integration:

1. If the limits of equation (1) are constants then this equation is called Fredholm integral equation. In this case, Fredholm integral equations of the first and second kinds will respectively have the following expressions, [3]:

$$f(x) = \int_{a}^{b} k(x,t)u(t)dt \qquad \ldots(4)$$

$$u(x) = f(x) + \int_{a}^{b} k(x,t)u(t)dt \qquad \ldots(5)$$

where a, b are constants.

2. If the upper limit of the integration in equation (1) is a variable then equation (1) is called Volterra integral equation. They are divided into two groups referred to as the first and second kinds.

Volterra integral equation of the first kind is,[3]:-

$$f(x) = \int_a^x k(x,t)u(t)dt \qquad \ldots(6)$$

and Volterra integral equation of the second kind is:-

$$u(x) = f(x) + \int_a^x k(x,t)u(t)dt \qquad \ldots(7)$$

where a is constant and x is variable.

The integral equation (1) is said to be singular if the range of integration is infinite $0 < x < \infty$ or $-\infty < x < \infty$, or if the kernel k(x, t) is unbou nded.

If the kernel k(x, t) in equation (1) depends only on the difference x-t, such a kernel is called a difference kernel and the equation:

$$a(x)u(x) - \int_\Omega k(x,t)u(t)dt = f(x) \qquad \ldots(8)$$

is called integral equation of convolution type.

2. The Decomposition method Applied to Fredholm integral Equation

In this subsection a Decomposition method to find the approximation solution for Fredholm integral equation is discussed [5],[6],[7].

Let us reconsider the Fredholm integral equation of the second kind.

$$f(t) = g(t) + \int_a^b k(t,s)f(s)ds, \qquad \ldots(9)$$

where g and the kernel k are known functions and f is the unknown function to be determined.

From equation (9), we obtain canonical form of Adomian's equation by writing

$$f(t) = g(t) + N(t) \qquad \ldots(10)$$

where

$$N(t) = \int_a^b k(t,s) f(s) ds \qquad \ldots(11)$$

To solve by Adomian's method, let

$f(t) = \sum_{m=0}^{\infty} f_m(t)$, and $N(t) = \sum_{m=0}^{\infty} A_m$ where A_m, $m = 0,1,\ldots,$ are polynomials depending on $f_0, f_1, \ldots, f_m$ and they are called Adomian polynomials. Hence, (10) can be rewritten as:

$$\sum_{m=0}^{\infty} f_m(t) = g(t) + \sum_{m=0}^{\infty} A_m(f_0, f_1, \ldots, f_m). \qquad \ldots(12)$$

From (11) we define:

$$\begin{cases} f_0(t) = g(t), \\ f_{m+1}(t) = A_m(f_0, f_1, \ldots, f_m), \qquad m = 0,1,2,\ldots \end{cases} \qquad \ldots(13)$$

In practice, all terms of the series $f(t) = \sum_{m=0}^{\infty} f_m(t)$ can not be determined and so we use an approximation of the solution by the following truncated series:

$$\varphi_k(t) = \sum_{m=0}^{k-1} f_m(t), \qquad \text{with } \lim_{k \to \infty} \varphi_k(t) = f(t). \qquad \ldots(14)$$

To determine Adomian polynomials, we consider the expansions:

$$f_\lambda(t) = \sum_{m=0}^{\infty} \lambda^m f_m(t), \qquad \ldots(15)$$

$$N_\lambda(f) = \sum_{m=0}^{\infty} \lambda^m A_m, \qquad \ldots(16)$$

Where, λ is a parameter introduced for convenience. From (16) we obtain:

$$A_m = \frac{1}{m!}\left[\frac{d^m}{d\lambda^m} N_\lambda(f)\right]_{\lambda=0}, \qquad \ldots(17)$$

and from (11), (15) and (17) we have:

$$A_m(f_0, f_1, \ldots, f_m) = \int_a^b v(s,t)\left[\frac{1}{m!}\frac{d^m}{d\lambda^m}\sum_{l=0}^{\infty}\lambda^l f_l\right]_{\lambda=0} ds$$

$$= \int_a^b v(s,t) f_m ds. \qquad \ldots(18)$$

So, (13) for the solution of linear Fredholm integral equation will be as follow:

$$\begin{cases} f_0(t) = g(t), \\ f_{m+1}(t) = \int_a^b v(s,t) f_m(t) ds, \qquad \text{m} = 0,1,2, \ldots \end{cases} \qquad \ldots(19)$$

Considering (13), we obtain:

$$\varphi_k(t) = g(t) + \int_a^b k(t,s) f_m(s) ds, \qquad \text{m} = 0,1,2,\ldots \qquad \ldots(20)$$

In fact (15) is exactly the same as the well known successive approximations method for solving of linear Fredholm integral equation defining as, [4]:

$$f_{m+1}(t) = g(t) + \int_a^b k(t,s) f_m(s) ds, \qquad \text{m} = 0,1,2,\ldots \qquad \ldots(21)$$

The initial approximations for the successive approximation method is usually zero function. In the other words, if the initial approximation in this method is selected that is $f_0(t) = g(t)$, then the Adomian decomposition method and the successive approximation method are exactly the same

The following algorithm summarizes the steps for finding the approximation solution for the second kind of Fredholm integral equation.

3. Algorithm (ADFI)

Input: $(g(t), k(t,s), f(s), a, x)$;

Output: series solution of given equation

Step1:

Put $f_0(t) = g(t)$

Step2:

Compute $f_{m+1}(t) = \int_a^b k(x,t) f_m(s) ds \qquad m = 0,1,\dots$

Step3:

Find the solution $\varphi_k(t) = \sum_{m=0}^{k-1} f_m(t), \qquad m = 0,1,\dots$

End

Example(1):

Consider the following Fredholm integral equation of the second kind with the exact solution $f(t) = t$.

$$f(t) = \frac{2x}{3} + \int_0^1 xtf(t)dt.$$

where compare the Fredholm integral equation of the second kind, we total

$$g(t) = \frac{2t}{3} \qquad , \quad k(x,t) = xt$$

To derive the solution by using the decomposition method, we can use the following Adomian scheme:

$$f_0(t) = \frac{2t}{3},$$

And

$$f_{m+1}(t) = \int_0^1 xtf_m(t)dt, \qquad \text{m} = 0,1,2,\ldots$$

For the first iteration, we have:

$$f_1(t) = \int_0^1 xtf_0(t)dt,$$

Then $f_1(t) = \frac{2t}{9}$

Considering (14), the approximated solution with two terms are:

$$\varphi_2(t) = f_0(t) + f_1(t)$$

$$\varphi_2(t) = \frac{2t}{3} + \frac{2t}{9}$$

Next term is:

$$f_2(t) = \int_0^1 xtf_1(t)dt,$$

$$f_2(t) = \frac{2t}{27}$$

Solution with three terms are:

$$\varphi_3(t) = f_0(t) + f_1(t) + f_2(t),$$

$$\varphi_3(t) = \frac{2t}{3} + \frac{2t}{9} + \frac{2t}{27},$$

In the same way, the component $\varphi_k(t)$ can be calculated for $k = 3,4,\ldots$ The solution with twelve terms are given as:

$$\varphi_{12}(t) = f_0(t) + f_1(t) + f_2(t) + \ldots + f_{11}(t).$$

$$\varphi_{12}(t) = \frac{2t}{3} + \frac{2t}{9} + \frac{2t}{27} + \frac{2t}{81} + \frac{2t}{729} + \frac{2t}{6561} + \frac{2t}{19683} + \frac{2t}{59049} + \frac{2t}{177147} + \frac{2t}{731441} + \frac{2\mathrm{t}}{2194323} + \frac{2t}{6582969} + \frac{2t}{59246721}$$

Approximated solution for some values of t by using Decomposition method and exact values $f(t) = t$ of Example(1), depending on least square error (L.S.E) are presented in Table(1).

4. Conclusion

This paper presents the use of the Adomian decomposition method, for the Fredholm integral equation. As it can be seen, the Adomian decomposition method for Fredholm integral equation is equivalent to successive approximation method. Although, the Adomian decomposition method is a very powerful device for solving the integral equation. From solving a numerical example the following points have been identified:

1- this method can be used to solve the second kind of linear Fredholm integral equation.

It is clear that using the decomposition method basis function to approximate when the m order is increases the error will be decreases.

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Table (1) The results of Example(1) using **(ADFI)** algorithm.

t	$f(t)$ (Exact)	$\varphi_{12}(t)$ (Approximation)
0	0	0
0.1	0.100	0.0999081
0.2	0.200	0.199999
0.3	0.300	0.299999
0.4	0.400	0.399999
0.5	0.500	0.499999
0.6	0.600	0.599999
0.7	0.700	0.699999
0.8	0.800	0.799999
0.9	0.900	0.899999
1	1	1.0082274
L.S.E		0.000001